Dynamical Low-Rank Approximation for Plasma Kinetic Model Reduction

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Outline

What are plasmas?

Plasmas are ionized, diffuse gases composed of charged particles (electrons and ions), and subject to electromagnetic forces. They are hard to model, but even harder to understand without computational modeling.

Two applications of plasma modeling

Nuclear fusion requires taming a bewildering array of plasma instabilities.

62.

10.2. Name Index

In this index the number preceding the class symbol refers to the corresponding page in the Appendix.

Absolute instability:		see Section 3.4.2.1. (1)
Acoustic instabilities	39.	(PV'Cn'S'1'TB'E'LMR'A'I)
Acoustic wave instability in a partially ionized plasma	8.	(PV'CnSsTB'E'LMR'A'I)
Alfvén wave (fire hose) instability	57.	(PV'C'SsT'Bbf'ELM't'R'A'I)
Amplitude dispersion instability of whistlers	105.	(PV'C'S'1TBb'f'ELMR'AI)
Amplitude dispersion instabilities; relativistic modes	5.	(PVC'S'1TBb'ELMR'AI)
Anisotropic temperature instability:		see velocity space instability
Ballooning instability	42.	(PV'C'SsTBbf'ELMR'A'I)
Beam-centrifugal instability	73.	(PV'C'Ss'vTBb'f'E'LMRpA'I)
Beam-plasma instability:		see two-stream instability
Bernstein-Greene-Kruskal wave instability	111.	(PV'C'S'1TB'E'L'M'tRp'A'I)
Buckling instability	43.	(PV'C'SsTBbf'ELMR'A'I)
Bulge instability:		see sausage instability
Bunching instability:		see two-stream instability
Buneman instability:		see two-stream instability
Centrifugal instability:		see flute, Rayleigh-Taylor, and gravitation instabilities
Čerenkov instability	2.	(PVC'Ss'vTBb'f'ELMRpA'I)
Collective instability:		see Sector 3.4.4.
Collective electrostatic instabilities in a two-dimensional field	99.	(PV'C'STBb'E'L'M'A'I)
Collisional drift instability:		see drift-dissipative instability
Collisionless gravitation instability:		see gravitation instability; collisionless modes
Collisionless tearing instability:		see tearing instability; collisionless modes

Figure: Source [9]

Two applications of plasma modeling

Nuclear fusion requires taming a bewildering array of plasma instabilities.

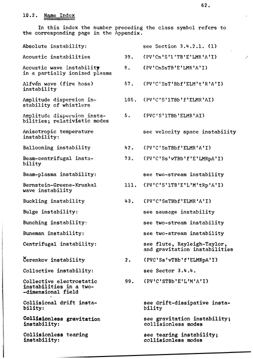

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Figure: Source [9]

Space is filled with plasma and magnetic fields.

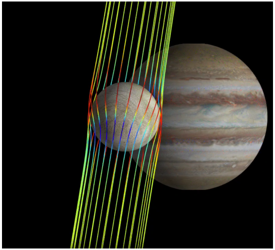

Figure: Europa's induction response to Jupiter's time-varying magnetic field provides strong evidence of a liquid ocean. [1]

The Boltzmann equation

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha + F \cdot \nabla_v f_\alpha = \left(\frac{\partial f_\alpha}{\partial t} \right)_{\text{coll}} = \sum_\beta S(f_\alpha, f_\beta).$$

- Indexed by particle species (α, β)
- $f_\alpha(x, v, t)$ a probability distribution
- $F = E + v \times B$ couples with Maxwell's equations
- Nonlinearity comes from field coupling and from $S(f_\alpha, f_\alpha)$.
- Posed in six dimensions!

The collision operator

...deserves its own slide.

Many collision models are used, but all satisfy a few invariants.

Important properties of $S(f_\alpha, f_\beta)$

- Conservation of mass, momentum, and energy:

$$\begin{bmatrix} \int f_\alpha \, dv_\alpha + \int f_\beta \, dv_\beta, \\ \int f_\alpha m_\alpha v_\alpha dv_\alpha + \int f_\beta m_\beta v_\beta \, dv_\beta, \\ \int f_\alpha m_\alpha \frac{|v_\alpha|^2}{2} \, dv_\alpha + \int f_\beta m_\beta \frac{|v_\beta|^2}{2} \, dv_\beta \end{bmatrix}$$

- Entropy growth: $\partial_t \int f \ln f \, dv \geq 0$ when for example $\partial_t f = S(f, f)$.

Together these drive the distribution function towards the Maxwellian distribution:

$$M(x, t) = \frac{n(x, t)}{(2\pi T(x, t))^{d/2}} e^{-\frac{|v - u(x,t)|^2}{2T(x,t)}},$$

Motivation for reduced models

- The kinetic equations are extremely costly to simulate directly.
  - For a 3D3V problem, 100 degrees of freedom in each dimension means 10^{12} grid points!
- Collisional diffusivity implies that hopefully not all those degrees of freedom are strictly needed in velocity space.
- Even much-reduced models can capture gross behavior, esp. flux of conserved quantities.

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Moment Methods

- Conservation laws posed only in physical space
- Includes classical fluid dynamics equations (Navier-Stokes, Euler)

Example: single species

Abbreviate the LHS by Df. We derive conservation of mass, momentum, and energy by integrating:

$$\langle Df = S(f,f), 1\rangle_v \implies \partial_t n + \nabla_x \cdot (nu) = 0,$$

$$\langle Df = S(f,f), v\rangle_v \implies$$

$$\partial_t (nu) + \nabla_x \cdot (nT\bar{\bar{I}} + nu \otimes u + \mathbb{P}) = n(E + u \times B)$$

$$\left\langle Df = S(f,f), |v|^2/2 \right\rangle_v \implies$$

$$\partial_t E + \nabla_x \cdot ((E + nT)u + \mathbb{P}u + \mathrm{q}) = E \cdot u$$

Fluid quantities are defined as[1]

$$n = \langle f \rangle_v, \quad u = \frac{1}{n}\langle fv \rangle_v,$$

$$E = \langle f|v|^2/2 \rangle_v, \quad T = \frac{1}{nd}\langle f|v-u|^2 \rangle_v.$$

[1] d is the dimension

Closure problem

Unfortunately, the moment system is not closed.

Problem moments

- Anisotropic pressure tensor (2 dimensions):

$$\mathbb{P} = \left\langle f \begin{bmatrix} v_x^2 - |v|^2/2 & v_x v_y \\ v_y v_x & v_y^2 - |v|^2/2 \end{bmatrix} \right\rangle_v ,$$

- Heat flux:

$$\mathrm{q} = \langle f |v - u|^2 (v - u) \rangle_v$$

...lead to the moment closure problem

The task is to come up with expressions for $\mathbb{P}, \mathrm{q}$ (or their gradients), in terms of the lower order moments.

Moment Closures

- Accomplishing a moment closure requires simplification. Count the degrees of freedom!
- Two main "families" of theories.
  - Chapman-Enskog theory: explicitly perturbative about a Maxwellian distribution.
  - Ansatz for f: assume f belongs to some class of distributions or satisfies some variational principle.

Caution

The chosen moment closure should result in a hyperbolic system. (Eigenvalues of flux matrix)

Chapman-Enskog theory

Introduce a small parameter by assuming collisions are strong: $\epsilon = Kn = \lambda / L$.

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha + F \cdot \nabla_v f_\alpha = \frac{1}{\epsilon} \sum_\beta S(f_\alpha, f_\beta)$$

Expand in powers of ϵ, about f_α^0:

$$f_\alpha = f_\alpha^0 + \epsilon f_\alpha^1 + \epsilon^2 f_\alpha^2 + \dots$$

Analyzing and retaining terms of a given order in ϵ results in familiar fluid equations:

- Order 0: Euler equations (neutral), ideal two-fluid equations (plasma). "Everything is magically Maxwellian".
- Order 1: Navier-Stokes equations (neutral), Braginskii closure (plasma)
- Order 2: Burnett equations (unstable!)

Hermite polynomial expansion

Grad 1949 [6] used an expansion in Hermite polynomials,[2]

$$f = f^0 \sum_{k=0}^{\infty} a^k \mathcal{H}^k \left(\frac{v-u}{v_{th}} \right)$$

- $\mathcal{H}^k$ are the tensorial rank-k Hermite polynomials,
- a^k a corresponding tensor of coefficients.
- $a^0 = 1, a^1_i = a^2_{ii} = 0$

Hermite polynomials?

The Hermite polynomials are orthogonal with respect to the Gaussian (Maxwellian) weight:

$$\int H^i(v) H^j(v) e^{-v^2/2}\, dv = \sqrt{2\pi} j! \delta_{ij}$$

Idea is to truncate the expansion.

[2] v_{th} is the thermal velocity, proportional to $\sqrt{T}$.

Moment methods recap

Benefits

- Capture the collisional asymptotic limit
- Reproduce conservation laws
- Focus computing resources on the quantities we care about: Grad's "physically meaningful" moments.
- Very natural to discretize using DG or finite volume methods.

Drawbacks

- Perturbative: moment convergence falls off quickly when far from a Maxwellian.
- Non-adaptive: no indication from the moments when your convergence is getting worse.

Brief history of dynamical low-rank methods

The dynamical low-rank method is a totally different direction for reduced plasma models.

- Proposed by Koch and Lubich for matrix-valued ODEs [7]
- Extended to higher-order tensors in various formats [8], [11]
- Overapproximation-insensitive integrators developed [10], [2]
- Applied to kinetic equations [5], [3]

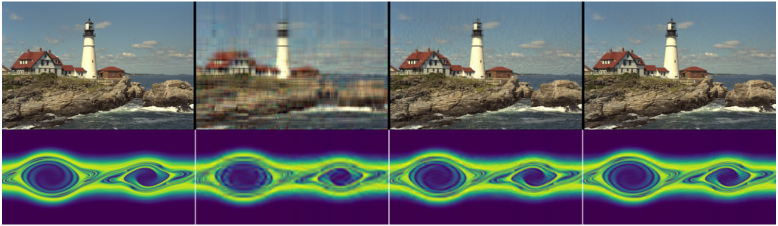


Figure: If the Singular Value Decomposition can compress a picture of a lighthouse, why not a phase space function?

Example low-rank kinetic approximation

Consider an arbitrary equation of "kinetic type":

$$\partial_t f(x, v, t) = h$$

Look for approximate solutions of the form

$$\tilde{f} = \sum_{i,j=1}^{r} X_i(x, t) S_{ij}(t) V_j(v, t).$$

- The integer r is the rank
- Bases X and V are orthonormal:

$$\langle X_i, X_k \rangle_x = \delta_{ik}, \quad \langle V_j, V_l \rangle_v = \delta_{jl}$$

- Similar to the SVD, except that S is not necessarily diagonal.
  - $\implies$ the decomposition is not unique.

What makes it dynamical?

We impose a Galerkin condition on the time derivative of the system:

$$\langle h - \tilde{h}, \delta \tilde{f} \rangle = 0 \quad \text{for all} \quad \delta \tilde{f} \in \mathcal{T}_{\tilde{f}} \mathcal{M}_r.$$

- $\mathcal{M}_r$ is the manifold of low-rank functions
- $\mathcal{T}_{\tilde{f}} \mathcal{M}_r$ is the tangent space to the manifold at $\tilde{f}$.
- $\tilde{h}$ is our low-rank time derivative.

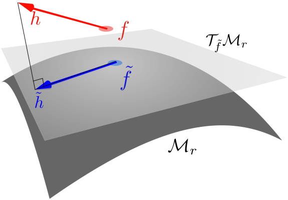


Figure: The method takes steps along the low-rank manifold's tangent space.

Projector-splitting integrator

Equivalently, $\tilde{h} = P(\tilde{f})h$ for an orthogonal projection $P(\tilde{f})$. It turns out that P has the form

$$\tilde{h} = P(\tilde{f})h = X_i \langle X_i, h \rangle_x - X_i \langle X_i V_j, h \rangle_{xv} V_j + \langle V_j, h \rangle_v V_j.$$

An operator splitting of the projector leads to a simple first order time integration scheme:

$$\begin{aligned} \tilde{f}' &= \tilde{f}(t_0) + \Delta t (V_j \langle V_j, h \rangle_v) \\ \tilde{f}'' &= \tilde{f}' - \Delta t (X_i \langle X_i V_j, h \rangle_{xv} V_j) \\ \tilde{f}(t_1) &= \tilde{f}'' + \Delta t (\langle X_i, h \rangle_x X_i). \end{aligned}$$

The scheme is:

- Exact if f has rank r.
- Robust to overapproximation: S can have vanishing singular values!

Analysis

Time-complexity savings can be huge.

$$r \ll N_x, N_v \quad \Longrightarrow \quad r^2(N_x + N_v) \ll N_x N_v$$

[3] Einkemmer ([4]) presents a conservative low-rank scheme.

Analysis

Time-complexity savings can be huge.

$$r \ll N_x, N_v \quad \Longrightarrow \quad r^2(N_x + N_v) \ll N_x N_v$$

Drawbacks

While an appealing technique, the dynamical low-rank method is not perfect.

- It fails to preserve mass, momentum, and energy.[3]
  - The approximation can "leak" conserved quantities into the truncated ranks.
- It fails to recover the asymptotic limit of the Maxwellian when collisions are strong.
  - This is because a Maxwellian with spatially varying parameters is not low-rank!

---

[3] Einkemmer ([4]) presents a conservative low-rank scheme.

A simple collisional plasma model

Test case: high-field limit of the Vlasov-Ampere-Fokker-Planck system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{1}{\epsilon} E \cdot \nabla_v f = \frac{1}{\epsilon} \nabla_v \cdot (vf + \nabla_v f), \\ \partial_t E = -J \end{cases}$$

- The current density J is defined $J = \langle vf \rangle_v$.
- Physically, can describe high-frequency motion of electrons against a static ion background fluid.

Limiting distribution

Define the local "Maxwellian"

$$M(x, v, t) = e^{-\frac{|v - E(x,t)|^2}{2}}$$

Then our Vlasov-Fokker-Planck equation is

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon} \nabla_v \cdot [M \nabla_v (M^{-1} f)].$$

As $\epsilon \to 0$, f approaches

$$f = \frac{\rho(x)}{(2\pi)^{d/2}} M,$$

for the density $\rho(x) = \langle f \rangle_v$.

Limiting fluid system

What is the governing equation for ρ in the $\epsilon \to 0$ limit?

- Multiply by 1 and v and integrate:

$$\partial_t \rho + \nabla_x \cdot J = 0$$

$$\partial_t J + \nabla_x \cdot \langle v \otimes v f \rangle_v = \frac{1}{\epsilon}(\rho E - J).$$

- Send $\epsilon \to 0$:

$$\rho E - J = 0 \quad \Longrightarrow \quad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho E) = 0, \\ \partial_t E = -\rho E. \end{cases}$$

The Asymptotic-preserving property

$$\begin{cases} C^{\text{kinetic}}: & \partial_t f + v\cdot\nabla_x f = \frac{1}{\epsilon}\nabla_v\cdot[M\nabla_v(M^{-1}f)], \\ C^{\text{fluid}}: & f = \frac{\rho}{(2\pi)^{d/2}}M, \text{ where } \partial_t\rho + \nabla_x\cdot(\rho E) = 0 \end{cases}$$

Both coupled with Ampere's equation.

We're looking for a kinetic discretization D^{kinetic} that makes this diagram commute:

$$\begin{tikzcd} D^{\text{kinetic}} \arrow[r, "\Delta t{,}x{,}v\to 0;\ r\to\infty"] \arrow[d, "\epsilon\to 0"] & C^{\text{kinetic}} \arrow[d, "\epsilon\to 0"] \\ D^{\text{fluid}} \arrow[r, "\Delta t{,}x{,}v\to 0"] & C^{\text{fluid}} \end{tikzcd}$$

Goal: the bottom path should hold for fixed r.

Low-rank algorithm

Key idea: evolve low-rank representation of $g = M^{-1} f$:

$$\epsilon \to 0 \quad \implies \quad g \to \frac{\rho(x)}{(2\pi)^{d/2}} \text{ (rank 1)}$$

Recall the low-rank ansatz is

$$g(x, v, t) = \sum_{i,j=1}^{r} X_i(x, t) S_{ij}(t) V_j(v, t).$$

Plug $f = gM$ into the Vlasov-Fokker-Planck equation:

$$h \triangleq \partial_t g = -v \cdot \nabla_x g - \frac{1}{M}(\partial_t M + v \cdot \nabla_x M) g + \frac{1}{\epsilon}[(\nabla_v - v + E) \cdot \nabla_v g].$$

Projector-splitting integration

Introduce auxiliary bases K and L:

$$K_j(x,t) = \sum_i X_i(x,t) S_{ij}(t), \quad L_i(v,t) = \sum_j S_{ij}(t) V_j(v,t).$$

1. Advance K_j^n to K_j^{n+1} via $\partial_t K_j = \langle V_j, h \rangle_v$.
2. Perform a QR decomposition of K_j^{n+1} to obtain X^{n+1}, S'.
3. Advance S' to S'' via $\partial_t S_{ij} = -\langle X_i V_j, h \rangle_{xv}$.
4. Advance L_i^n to L_i^{n+1} via $\partial_t L_i = \langle X_i, h \rangle_x$.
5. Perform a QR decomposition of L_i^{n+1} to obtain V^{n+1}, S^{n+1}.

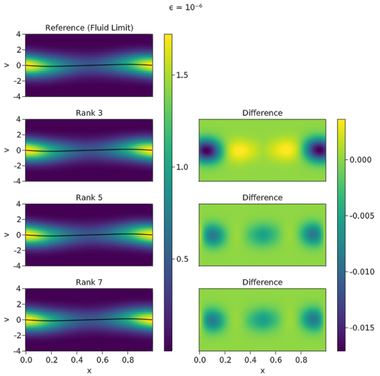
Spatial and temporal discretization

- First-order finite difference discretization in X and V
- First-order timestepping with implicit treatment of stiff collision terms
- Caution is required with collision term in "backwards" S step. Instability in time-reversed diffusion equation can be partly tamed with Forward Euler cancellation. E.g. $dy/dt = \lambda t$:

$$\begin{aligned} y_1 &= (1 - \lambda \Delta t)^{-1} y^n \\ y_2 &= (1 + \lambda(-\Delta t)) y_1 = y^n \\ y^{n+1} &= (1 - \lambda \Delta t)^{-1} y_2 = (1 - \lambda \Delta t)^{-1} y^n. \end{aligned}$$

Small-ϵ case

For very small ϵ, we capture the high field limit using only a small rank.



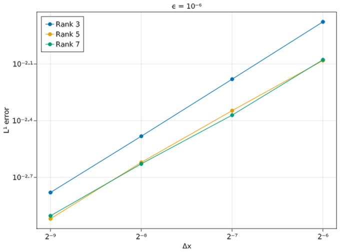


Figure: First-order convergence to the fluid limit in Δx

Singular value evolution

The singular values of the solution relax in one time step to order ϵ

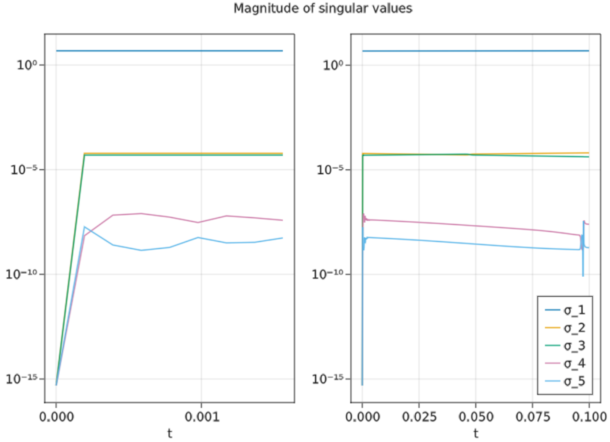


Figure: Singular values of solution beginning in local equilibrium

Singular value evolution

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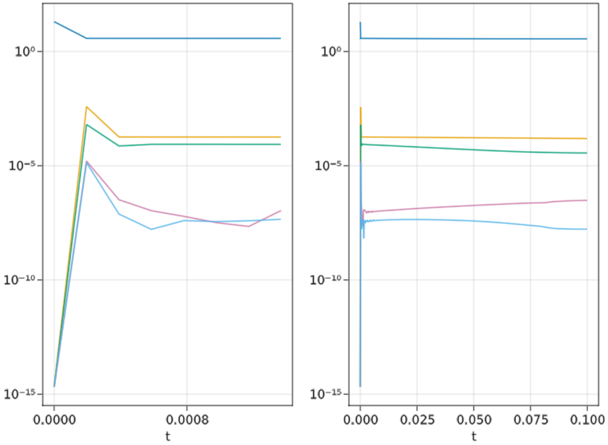

Figure: Singular values of solution beginning far from local equilibrium

Ratio of singular values

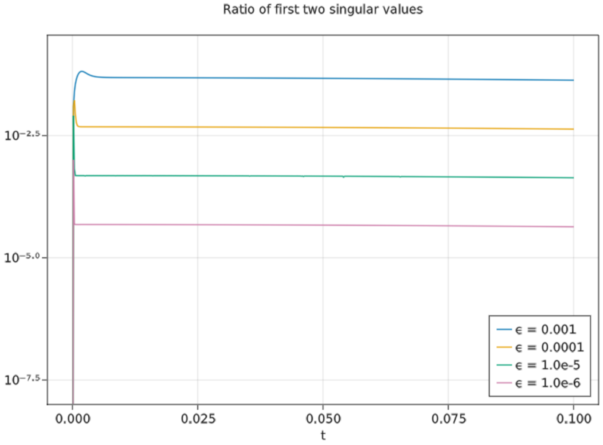

Figure: The ratio of the two largest singular values relaxes to order ϵ in one timestep.

Larger ϵ requires higher rank

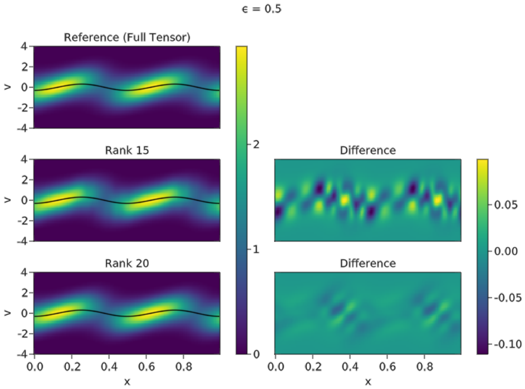

Figure: Higher rank is required to resolve less collisionless dynamics.

Conclusion

- Dynamical low-rank methods can capture the collisional fluid limit of kinetic equations.
- Arguably more flexible than moment methods, but less structure-preserving.

Next steps

- Application to magnetized, multispecies plasmas.
- Attempt to enforce conservation following Einkemmer [4].
- Introduce adaptivity by increasing rank as necessary.

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